Quasideterminants and Casimir elements for the general linear Lie superalgebra

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Abstract

We apply the techniques of quasideterminants to construct new families of Casimir elements for the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ whose images under the Harish-Chandra isomorphism are respectively the elementary, complete and power sums supersymmetric functions.

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1 Introduction

Let A be a square matrix over a ring. Its *quasideterminants* are certain rational expressions in the entries of A. The theory of quasideterminants originates from the papers by Gelfand and Retakh [2, 3] and since then a number of applications of the theory has been found; see [4] for an overview. In particular, the techniques of quasideterminants is fundamental in the theory of noncommutative symmetric functions developed by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [1]. The symmetric functions associated with a matrix whose entries are elements of a noncommutative ring is one of the interesting specializations of the general theory. When applied to the matrix E formed by the generators of the general linear Lie algebra $\mathfrak{gl}(n)$ the theory produces a new family of Casimir elements for $\mathfrak{gl}(n)$ as well as a distinguished set of generators of the Gelfand–Tsetlin subalgebra of $\mathrm{U}(\mathfrak{gl}(n))$; see [1, Section 7.4]. These results were extended to the orthogonal and symplectic Lie algebras in [8] with the use of the twisted Yangians and quantum determinants; see also a review paper [10].

In this paper we use the techniques of quasideterminants to get new families of Casimir elements for the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ and calculate their images with respect to the Harish-Chandra isomorphism. They can be regarded as super-analogs of those constructed in [1, Section 7.4]. Three families of Casimir elements are given explicitly in terms of some oriented graphs associated with $\mathfrak{gl}(m|n)$. The Harish-Chandra images turn out to be respectively the elementary, complete and power sums supersymmetric functions.

The starting point for our construction is a result of Nazarov [12]. He produced a formal series $B(t)$ called quantum Berezinian with coefficients in the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(m|n))$. Our first result is a quasideterminant factorization of $B(t)$ (Theorem 3.1). We then use it to get graph presentations for the Casimir elements (Theorem 4.1).

Some other families of Casimir elements for $\mathfrak{gl}(m|n)$ were constructed e.g. in [9]. This work is a super-version of the earlier constructions of [13, 14] for $\mathfrak{gl}(n)$ and it provides a linear basis of the center of $\mathrm{U}(\mathfrak{gl}(m|n))$ formed by the so-called quantum immanants.

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2 Preliminaries

Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ be two families of variables. A polynomial P in x and y is called supersymmetric if P is symmetric separately in x and y and satisfies the following cancellation property: the result of setting $x_m = -y_n = z$ in P is independent of z. We denote by $\Lambda(m|n)$ the algebra of supersymmetric polynomials in x and y. The algebra $\Lambda(m|n)$ is generated by the polynomials

$$p_k = x_1^k + \dots + x_m^k + (-1)^{k-1}(y_1^k + \dots + y_n^k), \qquad k \geq 1, \tag{2.1}$$

called the *power sums supersymmetric functions*. Two other families of generators of $\Lambda(m|n)$ are comprised by the *elementary* and *complete* supersymmetric functions defined respectively by the formulas

$$\begin{aligned} e_k &= \sum_{p+q=k} \sum_{i_1<\dots<i_p} \sum_{j_1\leq\dots\leq j_q} x_{i_1} \cdots x_{i_p} \, y_{j_1} \cdots y_{j_q}, \\ h_k &= \sum_{p+q=k} \sum_{i_1\leq\dots\leq i_p} \sum_{j_1<\dots<j_q} x_{i_1} \cdots x_{i_p} \, y_{j_1} \cdots y_{j_q}; \end{aligned} \tag{2.2}$$

see [15], [16].

We shall denote by E_{ij}, $i, j = 1, \dots, m+n$ the standard basis of the Lie superalgebra $\mathfrak{gl}(m|n)$. The $\mathbb{Z}_2$-grading on $\mathfrak{gl}(m|n)$ is defined by $E_{ij} \mapsto \bar{\imath} + \bar{\jmath}$, where $\bar{\imath}$ is an element of $\mathbb{Z}_2$ which equals 0 or 1 depending on whether $i \leq m$ or $i > m$. The commutation relations in this basis are given by

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}. \tag{2.3}$$

Given a $m+n$-tuple $(\lambda|\mu) = (\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n) \in \mathbb{C}^{m+n}$ we consider a highest weight $\mathfrak{gl}(m|n)$-module $L(\lambda|\mu)$ with the highest weight $(\lambda|\mu)$. That is, $L(\lambda|\mu)$ is generated by a nonzero vector ξ such that

$$\begin{aligned} E_{ii}\, \xi &= \lambda_i\, \xi && \text{for} \quad i = 1, \dots, m, \\ E_{m+j,m+j}\, \xi &= \mu_j\, \xi && \text{for} \quad j = 1, \dots, n, \\ E_{ij}\, \xi &= 0 && \text{for} \quad 1 \leq i < j \leq m+n. \end{aligned} \tag{2.4}$$

Any element z of the center $\mathrm{Z}(\mathfrak{gl}(m|n))$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(m|n))$ acts in $L(\lambda|\mu)$ as a scalar $\chi(z)$. For a fixed z the scalar $\chi(z)$ is a polynomial in λ_i and μ_i which is supersymmetric in the shifted variables defined by

$$\begin{aligned} x_i &= \lambda_i - i + 1 && \text{for} \quad i = 1, \dots, m, \\ y_j &= \mu_j + m - j && \text{for} \quad j = 1, \dots, n. \end{aligned} \tag{2.5}$$

Furthermore, the map $z \mapsto \chi(z)$ defines an algebra isomorphism

$$\chi : \mathrm{Z}(\mathfrak{gl}(m|n)) \to \Lambda(m|n), \tag{2.6}$$

which is called the Harish-Chandra isomorphism; see [5], [15], [16].

3 Decomposition of the Quantum Berezinian

Introduce the super-matrix $\widehat{E}$ of size $(m+n)\times(m+n)$ whose ij-th entry is $\widehat{E}_{ij} = (-1)^{\bar{j}} E_{ij}$. By the *quantum Berezinian* we mean the formal series $B(t)$ defined by

$$\begin{aligned} B(t) = \sum_{\sigma\in S_m} &\operatorname{sgn}\sigma\, \big(1+t\,\widehat{E}\big)_{\sigma(1),1}\cdots\big(1+t\,(\widehat{E}-m+1)\big)_{\sigma(m),m} \\ \times \sum_{\tau\in S_n} &\operatorname{sgn}\tau\, \big(1+t\,(\widehat{E}-m+1)\big)^{-1}_{m+1,m+\tau(1)}\cdots\big(1+t\,(\widehat{E}-m+n)\big)^{-1}_{m+n,m+\tau(n)}. \end{aligned} \tag{3.1}$$

The quantum Berezinian was constructed by Nazarov [12]. He also proved that all its coefficients are central in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(m|n))$. The image of $B(t)$ under the Harish-Chandra isomorphism is given by

$$\chi\big(B(t)\big) = \frac{(1+tx_1)\cdots(1+tx_m)}{(1-ty_1)\cdots(1-ty_n)}, \tag{3.2}$$

cf. [9]. Our first result is a decomposition of $B(t)$ into a product of quasideterminants. If X is a square matrix over a ring with 1 such that there exists the inverse matrix X^{-1} and its ji-th entry $(X^{-1})_{ji}$ is an invertible element of the ring, then the *ij-th quasideterminant of* X is defined by the formula

$$|X|_{ij} = \big((X^{-1})_{ji}\big)^{-1},$$

see [2, 3] for other equivalent definitions of the quasideterminants and their properties.

Theorem 3.1. *We have the following decomposition of $B(t)$ in the algebra of formal series with coefficients in* $\mathrm{U}(\mathfrak{gl}(m|n))$

$$\begin{aligned} B(t) = &\big|1+t\,\widehat{E}^{(1)}\big|_{11}\cdots\big|1+t\,(\widehat{E}^{(m)}-m+1)\big|_{mm} \\ &\times\big|1+t\,(\widehat{E}^{(m+1)}-m+1)\big|^{-1}_{m+1,m+1}\cdots\big|1+t\,(\widehat{E}^{(m+n)}-m+n)\big|^{-1}_{m+n,m+n}, \end{aligned} \tag{3.3}$$

where $\widehat{E}^{(k)}$ denotes the submatrix of $\widehat{E}$ corresponding to the first k rows and columns. Moreover, the factors in the decomposition are pairwise permutable.

Proof. We employ a quasideterminant decomposition of the quantum determinant for the Yangian $\mathrm{Y}(\mathfrak{gl}(r))$. The latter is the associative algebra with the generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1\le i,j\le r$ and the following defining relations

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}\big(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)\big), \tag{3.4}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}(n))[[u^{-1}]]. \tag{3.5}$$

Consider the quantum determinant of the matrix $T(u) = \big[t_{ij}(u)\big]$ defined by the following equivalent formulas

$$\begin{aligned}\operatorname{qdet} T(u) &= \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \cdot t_{\sigma(1),1}(u)\cdots t_{\sigma(r),r}(u-r+1)\\ &= \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \cdot t_{1,\sigma(1)}(u-r+1)\cdots t_{r,\sigma(r)}(u).\end{aligned} \tag{3.6}$$

It is well-known that the coefficients of this series are algebraically independent generators of the center of the algebra $\mathrm{Y}(\mathfrak{gl}(r))$; see e.g. [11] for a proof. For $1 \le k \le n$ denote by $T^{(k)}(u)$ the submatrix of $T(u)$ corresponding the first k rows and columns. We have the following quasideterminant decomposition of $\operatorname{qdet} T(u)$ in the algebra $\mathrm{Y}(\mathfrak{gl}(m))[[u^{-1}]]$

$$\operatorname{qdet} T(u) = |T^{(1)}(u)|_{11} \cdots |T^{(m)}(u-m+1)|_{mm}, \tag{3.7}$$

where the factors are pairwise permutable; see [8] and also [2], [6] for analogous decompositions in the case of noncommutative determinants of different types. Now we apply the algebra homomorphism $\mathrm{Y}(\mathfrak{gl}(m)) \to \mathrm{U}(\mathfrak{gl}(m|n))$ given by

$$T(u) \mapsto 1 + \widehat{E}^{(m)} u^{-1} \tag{3.8}$$

to (3.7), set $u = t^{-1}$ and multiply both sides by $(1-t)\cdots(1-(m-1)t)$. This will represent the first determinant factor in (3.1) as a product of quasideterminants which comprise the first m factors in (3.3); cf. [8].

Now consider the second factor in (3.1). We shall use the subscript (k) of a matrix to indicate its submatrix obtained by removing the first $k-1$ rows and columns. Here we need another version of the decomposition (3.7) given by

$$\operatorname{qdet} T(u) = |T_{(1)}(u-n+1)|_{11} \cdots |T_{(n)}(u)|_{nn}. \tag{3.9}$$

Apply another homomorphism $\mathrm{Y}(\mathfrak{gl}(n)) \to \mathrm{U}(\mathfrak{gl}(m|n))$ defined by

$$T(u) \mapsto \big[(1 + \widehat{E}\, u^{-1})^{-1}\big]_{(m+1)}, \tag{3.10}$$

(see [12]) to both sides of (3.9) with $\operatorname{qdet} T(u)$ expanded by the second formula in (3.6). Now observe that by the Inversion Theorem for quasiminors [2, 3], we have for any $k \in \{1, \dots, n\}$

$$\Big|\big[(1 + \widehat{E}\,(u-n+k)^{-1})^{-1}\big]_{(m+k)}\Big|_{m+k,m+k} = \big|1 + \widehat{E}^{(m+k)}\,(u-n+k)^{-1}\big|^{-1}_{m+k,m+k}. \tag{3.11}$$

To complete the argument, it remains to set $u = t^{-1} + n - m$ and divide both sides of the relation by the product $(1 + t(1 - m)) \cdots (1 + t(n - m))$.

Finally, note that the product of the first $m+n-1$ factors in (3.3) coincides with the quantum Berezinian for the subalgebra $\mathfrak{gl}(m|n-1)$ of $\mathfrak{gl}(m|n)$. Therefore the last factor in (3.3) is permutable with the elements of $\mathfrak{gl}(m|n-1)$ by the centrality of the quantum Berezinian. The proof is completed by an obvious induction. $\square$

4 Casimir elements

Let $A = (A_{ij})$ be a square matrix of size $l \times l$ with entries from an arbitrary ring and let t be a formal variable. Fix an integer i between 1 and l. Following [1, Definition 7.19] introduce the noncommutative symmetric functions associated with the matrix A and the index i as follows. The *elementary symmetric functions* $\Lambda_k^{(i)}$, the *complete symmetric functions* $S_k^{(i)}$, the *power sums symmetric functions of the first kind* $\Psi_k^{(i)}$ and the *power sums symmetric functions of the second kind* $\Phi_k^{(i)}$ are defined by the formulas

$$\begin{aligned}
1 + \sum_{k=1}^{\infty} \Lambda_k^{(i)}\, t^k &= |1 + tA|_{ii}, \\
1 + \sum_{k=1}^{\infty} S_k^{(i)}\, t^k &= |1 - tA|_{ii}^{-1}, \\
\sum_{k=1}^{\infty} \Psi_k^{(i)}\, t^{k-1} &= |1 - tA|_{ii}\, \frac{d}{dt}\, |1 - tA|_{ii}^{-1}, \\
\sum_{k=1}^{\infty} \Phi_k^{(i)}\, t^{k-1} &= -\frac{d}{dt} \log \big(|1 - tA|_{ii}\big).
\end{aligned} \tag{4.1}$$

These functions are polynomials in the entries of the matrix A and can be interpreted in terms of graphs in the following way. Let us consider the complete oriented graph $\mathcal{A}$ with l vertices $\{1, 2, \ldots, l\}$, the arrow from i to j being labelled by A_{ij}. Then every path in the graph going from i to j defines a monomial of the form $A_{ir_1} A_{r_1 r_2} \cdots A_{r_{k-1} j}$. A *simple path* is a path such that $r_s \neq i, j$ for every s. Then by [1, Proposition 7.20], $(-1)^{k-1}\Lambda_k^{(i)}$ is the sum of all monomials labelling simple paths in $\mathcal{A}$ of length k going from i to i; $S_k^{(i)}$ is the sum of all monomials labelling paths in $\mathcal{A}$ of length k going from i to i; $\Psi_k^{(i)}$ is the sum of all monomials labelling paths in $\mathcal{A}$ of length k going from i to i, where the coefficient of each monomial is the length of the first return to i; $\Phi_k^{(i)}$ is the sum of all monomials labelling paths in $\mathcal{A}$ of length k going from i to i, where the coefficient of each monomial is the ratio of k to the number of returns to i.

For any $i = 1, \dots, m$ consider the matrix $\widehat{E}^{(i)} - i + 1$ and the noncommutative symmetric functions associated with this matrix and the index i. We keep the above notation for these functions. Similarly, for any $j = 1, \dots, n$ consider the matrix $-\widehat{E}^{(m+j)} + m - j$ and the noncommutative symmetric functions associated with this matrix and the index $m + j$. Again, we denote the functions by the same symbols and distinguish them by the upper index $m + j$.

Theorem 4.1. *The algebra* $\mathrm{Z}(\mathfrak{gl}(m|n))$ *is generated by each of the families*

$$\begin{aligned}
\Lambda_k &= \sum_{i_1+\dots+i_{m+n}=k} \Lambda^{(1)}_{i_1} \cdots \Lambda^{(m)}_{i_m} \, S^{(m+1)}_{i_{m+1}} \cdots S^{(m+n)}_{i_{m+n}}, \\
S_k &= \sum_{i_1+\dots+i_{m+n}=k} S^{(1)}_{i_1} \cdots S^{(m)}_{i_m} \, \Lambda^{(m+1)}_{i_{m+1}} \cdots \Lambda^{(m+n)}_{i_{m+n}}, \\
\Psi_k &= \sum_{i=1}^{m} \Psi^{(i)}_k + (-1)^{k-1} \sum_{j=1}^{n} \Psi^{(m+j)}_k, \\
\Phi_k &= \sum_{i=1}^{m} \Phi^{(i)}_k + (-1)^{k-1} \sum_{j=1}^{n} \Phi^{(m+j)}_k,
\end{aligned} \tag{4.2}$$

where $k = 1, 2, \dots.$ *Moreover,* $\Psi_k = \Phi_k$ *for any* k, *and the Harish-Chandra images of these generators are respectively the elementary, complete and power sums supersymmetric functions,*

$$\chi(\Lambda_k) = e_k, \qquad \chi(S_k) = h_k, \qquad \chi(\Psi_k) = p_k. \tag{4.3}$$

Proof. Introduce the generating functions for the supersymmetric polynomials (2.1) and (2.2) by

$$\begin{aligned}
p(t) &= \sum_{k=1}^{\infty} p_k \, t^{k-1}, \\
e(t) &= 1 + \sum_{k=1}^{\infty} e_k \, t^k, \\
h(t) &= 1 + \sum_{k=1}^{\infty} h_k \, t^k.
\end{aligned} \tag{4.4}$$

These functions are related by

$$h(t) = e(-t)^{-1}, \qquad p(t) = -\frac{d}{dt} \log e(-t) = e(-t) \frac{d}{dt} \, e(-t)^{-1}, \tag{4.5}$$

see e.g. [7]. On the other hand, by Theorem 3.1 we have

$$1 + \sum_{k=1}^{\infty} \Lambda_k \, t^k = B(t) \tag{4.6}$$

which proves that the elements Λ_k are central in $\mathrm{U}(\mathfrak{gl}(m|n))$. Moreover, $\chi(B(t)) = e(t)$ due to (3.2) and so $\chi(\Lambda_k) = e_k$. The proof is completed by applying (4.5) and taking into account the fact that the factors in the decomposition (3.3) are mutually permutable; cf. the argument for the case of $\mathfrak{gl}(n)$ [1, Section 7.4]. $\square$

Example 4.2. We have

$$\begin{aligned}\Psi_1 &= \sum_{i=1}^{m}(E_{ii} - i + 1) + \sum_{j=1}^{n}(E_{m+j,m+j} + m - j),\\ \Psi_2 &= \sum_{i=1}^{m}\Big((E_{ii} - i + 1)^2 + 2\sum_{k=1}^{i-1} E_{ik}E_{ki}\Big)\\ &\quad - \sum_{j=1}^{n}\Big((E_{m+j,m+j} + m - j)^2 - 2\sum_{l=1}^{m+j-1}(-1)^{\bar{l}}E_{m+j,l}E_{l,m+j}\Big).\end{aligned} \tag{4.7}$$

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